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THE HOROCYCLIC RADON TRANSFORM ON NON-HOMOGENEOUS TREES*

BY

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ABSTRACT

This paper studies horocyles on trees and the corresponding Radon transformation. It is seen that a function can be reconstructed from the induced values on the horocycles. A formula is produced for the adjoint transformation and for the inverse.

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1. Introduction

The notion of Radon transform on $\mathbf{R}^2$, introduced by Radon [R] and further studied by John [J], both in the case of lines in the plane, has been extended to symmetric spaces, in particular to the hyperbolic disk, by Helgason [H]. There are two ways to extend the Radon transform to the disk, because there are two natural analogues of lines in hyperbolic geometry. The first is to geodesics, that is, circles in the hyperbolic disk which are orthogonal to the boundary. The corresponding integral transform is called the X-ray transform, in the terminology of [H]. The second is to horocycles, which give rise to the hyperbolic Radon transform.

Many recent papers have revealed a remarkable relationship between the hyperbolic disk and infinite trees (graphs without loops) and studied this analogy. When the hyperbolic disk is regarded as a homogeneous space under the action of its automorphism group $PSL(2,\mathbf{R})$, it is natural to compare it with a homogeneous tree, a tree each of whose vertices meets the same number of edges. On the other hand, when attention is focused upon real analysis or probability, the discrete counterpart of the disk is a generic infinite tree without any homogeneity conditions. A large part of the classical theory of Radon transforms fits within real analysis, and in this spirit X-ray transforms on trees have been studied in [BCCP], [CCP], [CC]. On the other hand, horocyclic Radon transforms have been investigated only on homogeneous trees: cf. [BP], [BFP]. These papers make use of radial functions and need a group of 'rotations' acting transitively on the circles around a fixed vertex of the tree, which is therefore assumed homogeneous.

The present paper considers the horocyclic Radon transform on a general tree T. After describing the geometry of horocycles in Section 2, we prove in Section 3 that the Radon transform is injective on $L^1(T)$, and give an expression for its inverse in terms of integral operators over the boundary of the tree. This general inverse is not explicit: in order to obtain explicit inversion formulas, we follow the usual procedure of introducing a dual Radon transform R^* (Section 4) and seeking the inverse of R^*R. This method does work for homogeneous trees, as proved in [BFP]: in Section 5 we give a simplified (but less elegant) proof of such a special case, which should hopefully adapt to the general setting. We are, however, unable to compute the inverse of R^*R for non-homogeneous trees. So we prove a different inversion formula (Section 6), which does not factor through the dual transform. This formula is new even in the homogeneous case, where it assumes a remarkably simple expression.

2. Horocycles

Let T be a connected tree. To avoid trivialities we shall henceforth assume that each vertex touches at least three edges, although most of the definitions and results set forth in the sequel extend to the general case. By a **path** we mean a sequence of vertices $[v_0, v_1, \dots]$ where $v_j \sim v_{j+1}$—that is, v_j and v_{j+1} are neighbors—and $v_j \neq v_{j+2}$ for all j. Unless otherwise specified, such a sequence will be assumed infinite. The boundary of T is the space Ω of **ends**, the classes of paths under the equivalence relation $\simeq$ generated by the unit shift: $[v_0, v_1, \dots] \simeq [v_1, v_2, \dots]$. If we fix a vertex u, then Ω can be identified with the set of paths beginning at u. Each end ω induces an orientation on the edges of T: an edge $[v, w]$ is positively oriented if there exists a representative path in ω which starts at v and contains w. For every ω and every pair of vertices v, w define the **horocycle index** $\kappa_\omega(v, w)$ as the number of positively oriented edges (with respect to ω) minus the number of negatively oriented edges in the path from v to w (which is unique since T is a tree). For $n \in \mathbf{Z}$, the set

$$h_u(n, \omega) = \{w \in T\colon \kappa_\omega(u, w) = n\}$$

is the **horocycle** of index n through ω with respect to u. The **distance** $d(v, w)$ between the vertices v, w is the length of the finite path between them.

Remark 2.1: The vertices of a horocycle all lie at even distance from each other. ∎

It is immediately seen that the family of horocycles through a fixed ω does not depend on the choice of the reference vertex u, but indices do: it would therefore be preferable to express horocycles in terms of paths instead of ends and vertices. If $p = [v_0, v_1, \dots]$ is a path, and w a vertex, let $d(w, p) = \min_{j \geq 0} d(w, v_j)$. Set

$$h_p = \{w \in T\colon d(w, v_0) = 2d(w, p)\} :$$

that is, w belongs to h_p if the nearest vertex of p is exactly halfway between w and v_0 (see Figure 2.1, where the vertices of h_p are circled: only a portion of the tree is depicted). Then h_p is the horocycle $h_{v_0}(0, \omega)$, where ω is the equivalence class of p. To see this, consider another equivalence relation on the set of paths:

if $p = [v_0, v_1, \dots]$, $p' = [v_0', v_1', \dots]$, we stipulate that $p \cong p'$ if $v_j = v_j'$ for every sufficiently large j. In this case $p \simeq p'$, so $\cong$ is strictly finer than $\simeq$. For the sake of completeness, we now prove the simple fact that h_p depends only on the $\cong$-equivalence class of p.

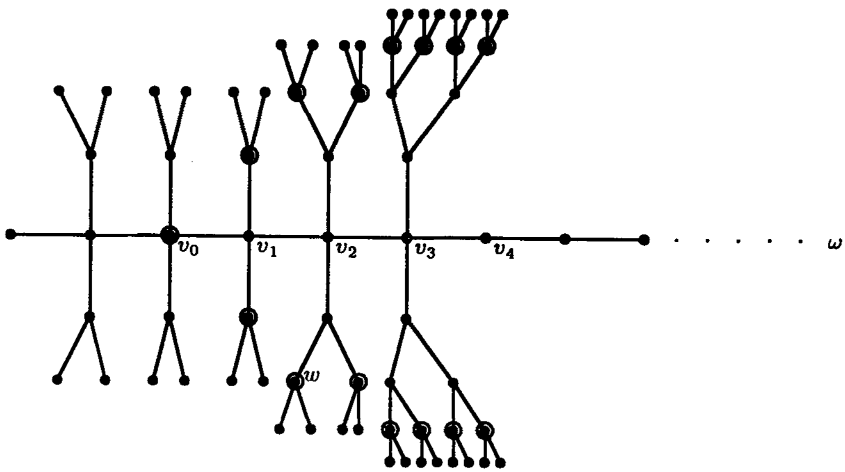

FIGURE 2.1.

PROPOSITION 2.2: *Let p, p' be paths. Then $h_p = h_{p'}$ if and only if $p \cong p'$. In particular, given a path p and $u \in h_p$ there exists a unique path p' starting at u and such that $h_p = h_{p'}$.*

Proof: Assume first that $v_j = v_j'$ exactly for $j \geq n$. Let $u \in h_p$ and assume that $d(u,p) = k$, so that the closest vertex of p to u is v_k and their distance is k. If $k \geq n$, then $v_k = v_k'$ is also the closest vertex of p' to u, thus $u \in h_{p'}$. Whereas if $k < n$, then the closest vertex of u to p' is v_n', and their distance is $k + (n-k) = n$, which is also the distance between v_n' and v_0', whence $v \in h_{p'}$.

Next assume that $h_p = h_{p'}$. Let $k = d(v_0, v_0')/2$ (recall Remark 2.1): since $v_0, v_0' \in h_p = h_{p'}$, we have $d(v_0, p') = k = d(v_0', p)$. Therefore u, the vertex which lies halfway between v_0 and v_0', belongs to both p and p', hence $v_k = u = v_k'$, while $v_j \notin p'$ and $v_j' \notin p$ for every $j < k$. Let m be the smallest index larger than k such that $v_m \notin p'$. Let w be a vertex at distance m from p, such that the closest vertex of p is v_m: then $w \in h_p$. Now, v_m lies between w and the vertex

$v_{m-1} = v'_{m-1}$, which is therefore the closest of p' to w, and $d(w,p') = m+1$. But $d(w,v'_0) = (m+1)+(m-1) = 2m \neq 2d(w,p')$, hence $w \notin h_{p'}$, which contradicts the assumption $h_p = h_{p'}$. Thus no such m exists, whence $v_j = v'_j$ for every $j \geq k$. ∎

Let $p = [v_0, v_1, \ldots]$, $p^- = [v_0, v_{-1}, \ldots]$ be paths starting at the same vertex v_0 but otherwise disjoint. For $n \in \mathbf{Z}$, define $h(n,p) = h_{[v_n, v_{n+1}, \ldots]}$. Obviously for $n \geq 0$ this depends only on n and (the $\cong$-equivalence class of) p. For $n < 0$ this apparently depends on the choice of p^-, but Proposition 2.2 shows that it does not.

Thus we have

PROPOSITION 2.3: *There is a one-to-one correspondence between $\mathbf{Z} \times \Omega$ and the set $\mathcal{H}$ of horocycles. This correspondence is not unique and depends on the choice of one vertex.*

Proof: Fix a vertex u. To any end ω and any integer n we can associate the horocycle $h_u(n,\omega) = h(n,p)$, where p is the unique representative path of ω which starts at u. This map is clearly injective. To see that it is also surjective, let h_p be a horocycle with $p = [v_0, v_1, \ldots]$, and let p' be the path beginning at u and representing the same end. Then $h_p = h(n,p')$, where $n = 2d(u,p) - d(u,v_0)$. ∎

Remark 2.4: Given a path $p = [v_0, v_1, \ldots]$, a vertex u is in the horocycle $h(n,p)$ for a unique value of n, namely $\kappa_p(u) = d(u,v_0) - 2d(u,p)$ (which equals $\kappa_\omega(v_0,u)$ if ω is the $\simeq$-equivalence class of p). Hence T is the disjoint union of the horocycles $h(n,p)$ for $n \in \mathbf{Z}$.

Definition 2.5: Let u, v be neighboring vertices. The set

$$S(u,v) = \{w \in T\colon d(w,u) = d(w,v) + 1\}$$

is called a **sector**. For general u, v set

$$S(u,v) = \{w \in T\colon d(w,u) = d(w,v) + d(u,v)\},$$

and notice that $S(u,u) = T$, while $S(u,v) = S(u',v)$, where u' is the neighbor of v lying between u and v if $u \neq v$.

On the other hand, we define a **wedge** as the set

$$W_p = \{w \in T\colon \kappa_p(w) \geq 0\},$$

where p is a path. So a wedge consists of all vertices w whose distance from v_0 is not smaller than twice the distance from p—in particular, W_p contains p (see Figure 2.2: the shaded region is the sector $S(u, v_0)$, while the vertices of the wedge W_p are circled). By Proposition 2.2, W_p only depends on the $\cong$-equivalence class of p, i.e., on the horocycle h_p. ∎

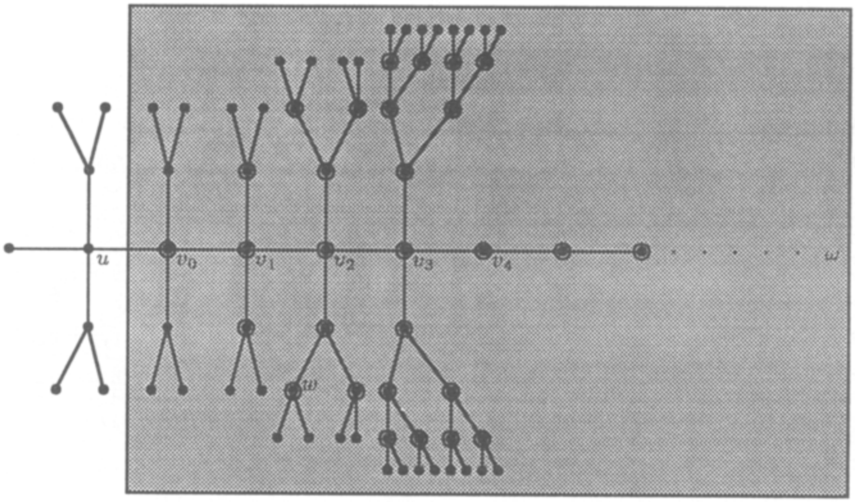

FIGURE 2.2.

Notice the disjoint union decompositions

$$T = S(u,v) \cup S(v,u) \quad \text{for every } u, v \in T \text{ with } u \sim v, \tag{2.1}$$

$$T = \{u\} \cup \bigcup_{v \sim u} S(u,v) \quad \text{for every } u \in T, \tag{2.2}$$

$$W_p = \bigcup_{j \geq 0} h(j,p) \quad \text{for every path } p. \tag{2.3}$$

PROPOSITION 2.6: *A sector can be decomposed into a disjoint union of horocycles.*

Proof: By (2.3) it will suffice to show that a sector $S(u, v)$, with $u \sim v$, can be decomposed as a disjoint union of wedges. Take a path $p = [v_0, v_1, \ldots]$ such that

$v_0 = v$ and $v_1 \neq u$. Then $W_p \subset S(u,v)$. We shall prove that

$$S(u,v) = W_p \cup \bigcup_{\substack{\kappa_p(w)=-1 \\ w \neq u}} S(v,w) \quad \text{(disjoint unions).} \tag{2.4}$$

Observe that the distance from u of each sector on the right-hand side is larger than that of $S(u,v)$.

If $z \in S(v,w)$ for some $w \neq u$ with $\kappa_p(w) = -1$, then necessarily the path connecting z to p contains w, hence

$$d(z,p) = d(z,w) + d(w,p),$$
$$d(z,v) = d(z,w) + d(w,v),$$

whence $\kappa_p(z) = \kappa_p(w) - d(z,w) \leq -1$, and $z \notin W_p$.

Conversely take $z \in S(u,v)$ such that $\kappa_p(z) \leq -1$. It is easy to see that the index κ_p increases by one at each step when moving from z towards p. Call w the vertex for which $\kappa_p(w) = -1$, and observe that w lies on the path between v and z, i.e., $z \in S(v,w)$.

Now (2.4) can be used to prove by induction that for every $n \geq 1$ there exist vertices $w_{n,1}, \ldots, w_{n,j_n} \in S(u,v)$ at distance n from v, and paths $p_{n,1}, \ldots, p_{n,k_n}$ such that

$$S(u,v) = \bigcup_{m=1}^{n} \bigcup_{i=1}^{k_m} W_{p_{m,i}} \cup \bigcup_{i=1}^{j_n} S(u, w_{n,i}) \quad \text{(disjoint unions).}$$

Since the distance of $S(u, w_{n,i})$ from v tends to infinity with n, in the limit we have

$$S(u,v) = \bigcup_{m=1}^{\infty} \bigcup_{i=1}^{k_m} W_{p_{m,i}} \quad \text{(disjoint unions).} \qquad \blacksquare$$

3. The inverse Radon transform as an integral operator

The (**horocyclic**) **Radon transform** R on T is defined as follows: if $f \in L^1(T)$ then $Rf \in L^\infty(\mathcal{H})$ is given by

$$Rf(h) = \sum_{v \in h} f(v) \quad \text{for every horocycle } h \in \mathcal{H}.$$

THEOREM 3.1: *The Radon transform is injective.*

Proof: Suppose $Rf \equiv 0$ on $\mathcal{H}$. For any pair of neighboring vertices u, v, Proposition 2.6 yields the disjoint union expression $S(u,v) = \bigcup_{j\in\mathbf{N}} h_j$, for suitable horocycles h_j. Then the integral of f over a sector vanishes:

$$\sum_{w\in S(u,v)} f(w) = \sum_{j\in\mathbf{N}} \sum_{w\in h_j} f(w) = \sum_{j\in\mathbf{N}} Rf(h_j) = 0.$$

Now it is enough to expand $\sum_{w\in T} f(w)$ in a similar way, using the disjoint union decompositions (2.1) on one side and (2.2) on the other side, to get $f(u) = 0$. ∎

The same argument can be used to prove a more general statement (the **support theorem**): if Rf vanishes on every horocycle which does not intersect a fixed subset K of T, then f vanishes outside K (cf. [BCCP, Theorem 1.3] for the X-ray transform case). The proof of Theorem 3.1—along with that of Proposition 2.6—also provides an algorithm for inverting the Radon transform. Observe that the above proof (as well as the inversion formula given in Theorem 6.1 below) only uses horocycles which have nonnegative index with respect to a fixed reference vertex.

We shall now express R^{-1} as an integral operator. The space $\mathcal{H}$ is equipped with a natural Borel structure, induced from the product topology of $\mathbf{Z}\times\Omega$ via the bijection of Proposition 2.3. Recall that the topology of Ω is defined as follows [C]: the sequence of ends (ω_j) tends to the end ω if there exist paths $p_j \in \omega_j$ for each $j \in \mathrm{N}$ and $p \in \omega$, all sharing the same initial vertex, such that p_j coincides with p for a number of steps $d_j \to \infty$. Denote by $\mathcal{H}_w$ the set of horocycles which contain $w \in T$, and by $\mathcal{H}_w^v$ the subset of $\mathcal{H}_w$ of those contained in $S(v,w)$ (in particular, $\mathcal{H}_v^v = \mathcal{H}_v$).

THEOREM 3.2: *For every $v \in T$ there exists a (signed) Borel measure ν_v on $\mathcal{H}$ such that*

$$\nu_v(\mathcal{H}_w) = \begin{cases} 1 & \text{if } v = w, \\ 0 & \text{if } v \neq w. \end{cases}$$

If $\{\nu_v : v \in T\}$ is a family of such measures then

$$f(v) = \int_{\mathcal{H}} Rf(h)\, d\nu_v(h) \quad \text{for every } v \in T.$$

Proof: It is clear that a family $\{\nu_v\}$ of measures that satisfy the above property inverts R. Indeed, it is enough to restrict attention to $f = \delta_w$, the Dirac function at w (i.e., the characteristic function of the set $\{w\}$ in T), for each $w \in T$. In this case $Rf = \chi_{\mathcal{H}_w}$, the characteristic function of $\mathcal{H}_w$ in $\mathcal{H}$, and

$$\int_{\mathcal{H}} Rf\, d\nu_v = \int_{\mathcal{H}} \chi_{\mathcal{H}_w}\, d\nu_v = \nu_v(\mathcal{H}_w) = \delta_w(v) = f(v) \quad \text{for every } v \in T.$$

The existence of such a family of measures is less obvious, because the sets $\{\mathcal{H}_v : v \in T\}$ are not disjoint. Fix $v \in T$. It is easy to see that $\mathcal{H}_v \cap \mathcal{H}_w$ is empty if $d(v,w)$ is odd. Thus we can set $\nu_v = 0$ on every Borel subset of

$$\bigcup_{w:\ d(v,w) \text{ is odd}} \mathcal{H}_w.$$

On the rest of $\mathcal{H}$ we shall construct ν_v on each $\mathcal{H}_w$ by induction on the (even) distance $d(v,w)$, starting with $\mathcal{H}_v$. Let $\nu_v|_{\mathcal{H}_v}$ be any Borel measure of mass 1 on $\mathcal{H}_v$, so that $\nu_v(\mathcal{H}_v) = 1$. Label all vertices of T at even distance from v as $v_0 = v,\ v_1,\ v_2,\ \ldots$ in a way that $d(v, v_j)$ is nondecreasing, and assume that ν_v has already been constructed on $\mathcal{H}_{(j)} = \bigcup_{j \le n} \mathcal{H}_{v_j}$ and satisfies the condition

$$\nu_v(\mathcal{H}_{v_j}) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } 0 < j \le n. \end{cases}$$

To perform the induction step observe that $\mathcal{H}_{v_{j+1}} \setminus \mathcal{H}_{(j)}$ is nonempty, because it contains $\mathcal{H}^v_{v_{j+1}}$. So we can set $\nu_v|_{\mathcal{H}_{v_{j+1}} \setminus \mathcal{H}_{(j)}}$ to be any Borel measure on $\mathcal{H}_{v_{j+1}} \setminus \mathcal{H}_{(j)}$ of mass $-\nu_v(\mathcal{H}_{v_{j+1}} \cap \mathcal{H}_{(j)})$, and consequently

$$\nu_v(\mathcal{H}_{v_{j+1}}) = \nu_v(\mathcal{H}_{v_{j+1}} \cap \mathcal{H}_{(j)}) + \nu_v(\mathcal{H}_{v_{j+1}} \setminus \mathcal{H}_{(j)}) = 0. \qquad \blacksquare$$

4. The dual Radon transform

Let $v \in T$, and for each vertex w let $q_w + 1$ be the number of neighbors of w. A probability measure ρ_v on Ω which arises naturally (cf. [BCCP, §2]), and which reduces to the unique rotation-invariant one if T is homogeneous, is determined by

$$\rho_v(\Omega^v_w) = \begin{cases} 1 & \text{if } v = w, \\ \dfrac{1}{q_{v_0} + 1} \displaystyle\prod_{j=1}^{n-1} \frac{1}{q_{v_j}} & \text{if } v \ne w, \end{cases}$$

where $[v_0{=}v, v_1, \ldots, v_n{=}w]$ is the finite path from v to w, and Ω^v_w is the space of ends determined by paths contained in $S(v,w)$. If Ω' is a Borel subset of

Ω, the quantity $\rho_v(\Omega')$ expresses the probability that a particle starting from v eventually hits Ω'. It is assumed that the particle follows a random path, at each move choosing any of its neighboring vertices except the previous with equal probability. The corresponding positive Borel measure μ_v on $\mathcal{H} = \mathbf{Z} \times \Omega$ is the product of the counting measure on $\mathbf{Z}$ and the above-defined measure ρ_v on Ω.

PROPOSITION 4.1: *We have:*

$$\mu_v(\mathcal{H}) = \infty, \tag{4.1}$$

$$\mu_v(\mathcal{H}') = \rho_v(\Omega') \quad \text{for every } w \in T \text{ and Borel subset } \mathcal{H}' \text{ of } \mathcal{H}_w, \tag{4.2}$$

where Ω' is the set of ends associated to the horocycles in $\mathcal{H}'$, and

$$\mu_v(\mathcal{H}_\omega) = 0 \quad \text{for every } \omega \in \Omega, \tag{4.3}$$

where $\mathcal{H}_\omega = \{h_v(n,\omega) : n \in \mathbf{Z}\}$ is the set of horocycles through the end ω. In particular, for every $w \in T$,

$$\mu_v(\mathcal{H}_w) = 1,$$
$$\mu_v(\mathcal{H}_w^v) = \rho_v(\Omega_w^v).$$

Proof: Formula (4.1) holds since $\rho_v(\Omega) = 1$ and $\mathcal{H} = \mathbf{Z} \times \Omega = \bigcup_{n\in\mathbf{Z}}\{n\} \times \Omega$.

To prove (4.2) observe that the projection of $\mathcal{H}_w$ onto the second factor in $\mathbf{Z} \times \Omega$ is one-to-one (in other words, there is only one horocycle in $\mathcal{H}_w$ which passes through each given end of T), and recall that the first factor is endowed with the counting measure.

A single end ω has ρ_v measure zero (because $q_w \geq 2$ for every $w \in T$): thus from $\mathcal{H}_\omega = \mathbf{Z} \times \{\omega\} = \bigcup_{n\in\mathbf{Z}}\{n\} \times \{\omega\}$ we get (4.3). ∎

Remark 4.2: The proof of Theorem 3.2 can be easily adapted so that the measure ν_v is absolutely continuous with respect to μ_v, for each $v \in T$. ∎

The **dual Radon transform** R^* on T is defined in terms of the chosen family $\{\mu_v : v \in T\}$ as follows: if $\phi \in L^\infty(\mathcal{H})$ then $R^*\phi \in L^\infty(T)$ is given by

$$R^*\phi(v) = \int_{\mathcal{H}_v} \phi(h)\, d\mu_v(h) \quad \text{for every } v \in T.$$

This transform is the adjoint of R in the sense that

$$\int_{\mathcal{H}} \phi\, R\delta_v\, d\mu_v = \int_{\mathcal{H}} \phi\, \chi_{\mathcal{H}_v}\, d\mu_v = \int_{\mathcal{H}_v} \phi\, d\mu_v = R^*\phi(v) = \sum_{w\in T} R^*\phi(w)\, \delta_v(w).$$

Observe that R^* only involves, for each v, the restriction of μ_v to $\mathcal{H}_v$.

The operator R^*R therefore maps $L^1(T)$ to $L^\infty(T)$. In order to find an integral representation for it, we first compute it on Dirac deltas. Given vertices v, w, we have

$$R^*R\delta_w(v) = \int_{\mathcal{H}_v} \chi_{\mathcal{H}_w}\, d\mu_v = \mu_v(\mathcal{H}_v \cap \mathcal{H}_w).$$

This quantity vanishes if v and w are an odd distance apart, because in this case $\mathcal{H}_v$ and $\mathcal{H}_w$ are disjoint by Remark 2.1. If the distance is nonzero and even, let $[v_0{=}v, v_1, \ldots, v_{2n}{=}w]$ be the finite path from v to w, and observe that the end ω corresponding to each horocycle h in $\mathcal{H}_v \cap \mathcal{H}_w$ must belong to $\Omega^v_{v_n} \cap \Omega^w_{v_n} = \Omega^v_{v_n} \setminus \Omega^v_{v_{n+1}}$. In fact the orientation of the edges $[v_j, v_{j+1}]$ must be positive with respect to ω for $0 \le j < n$, and negative for $n \le j < 2n$, in order for the index $\kappa_\omega(v,w)$ to be zero. Therefore

$$\mu_v(\mathcal{H}_v \cap \mathcal{H}_w) = \rho_v(\Omega^v_{v_n}) - \rho_v(\Omega^v_{v_{n+1}}) = \frac{q_{v_n} - 1}{(q_{v_0}+1)\prod_{j=1}^n q_{v_j}}$$

(for $d(v,w)$ nonzero even). Finally $\mu_v(\mathcal{H}_v) = 1$, as remarked before. For an arbitrary $f \in L^1(T)$, we thus have

PROPOSITION 4.3: *The operator $R^*R\colon L^1(T) \to L^\infty(T)$ has the integral representation*

$$R^*Rf = \sum_{w \in T} \psi(\,\cdot\,, w) f(w) \quad \text{for every } f \in L^1(T),$$

where the kernel ψ is given by

$$\psi(v,w) = \mu_v(\mathcal{H}_v \cap \mathcal{H}_w) = \begin{cases} 1 & \text{if } v = w, \\ 0 & \text{if } d(v,w) \text{ is odd}, \\ \dfrac{q_{v_n} - 1}{(q_{v_0}+1)\prod_{j=1}^n q_{v_j}} & \text{if } d(v,w) = 2n > 0. \end{cases}$$ ∎

5. Geometric inversion in the homogeneous setting

Let us examine the operator R^*R and its invertibility in the homogeneous case. The results in this section are adapted from [BFP].

Let T be homogeneous of degree $q+1 \ge 3$, i.e., each vertex w has exactly $q_w + 1 = q + 1$ neighbors. The tree T can in this case be identified with a group, namely the free product of $q+1$ copies of $\mathbf{Z}_2$. By Proposition 4.3, R^*R acts as the convolution operator with kernel

$$\psi(v,w)=\begin{cases}1 & \text{if } v=w,\\ 0 & \text{if } d(v,w) \text{ is odd},\\ \dfrac{q-1}{(q+1)q^n} & \text{if } d(v,w)=2n>0.\end{cases}$$

Note that ψ actually depends only on $d(v,w)$. Letting v be the vertex that represents the identity element of the group, the convolver can be regarded as the *function* $\Psi = \sum_{n=0}^{\infty} c_n\chi_{2n}$ on T, where χ_m is the characteristic function (on T) of the set $C(v,m)$ of vertices at distance m from v, and the coefficient c_n equals $\psi(v,w)$ for $d(v,w) = 2n$. Naturally Ψ is ***radial*** (with respect to v), that is, constant on $C(v,m)$ for each m. Because $C(v,m)$ consists of exactly $(q+1)q^{m-1}$ vertices if $m > 0$, one easily verifies that $\Psi \in L^{2+\epsilon}(T)$ if and only if $\epsilon > 0$. In particular, R^*R is not bounded as an endomorphism of L^1. Its inverse, therefore, must be found on a larger L^p. However, R^*R is bounded from L^1 to $L^{2+\epsilon}$ for every $\epsilon > 0$. A bounded inverse $J\colon L^{2+\epsilon} \to L^1$ would yield a more geometric and explicit inversion formula than the general integral representation of Theorem 3.2: if $\phi = Rf$, we would recover f as $JR^*\phi$. Unfortunately no such bounded J exists: since R^*R is a right convolution operator, J would have to commute with the left action of $\mathrm{Aut}(T)$, hence would be the convolution on the right with a radial function Φ. If the range of J were in L^1, the convolver Φ would also have to be in L^1. We shall compute Φ and show that it belongs to $L^{1+\epsilon}$ only for $\epsilon > 0$.

On the other hand, R^*R need only be inverted on its range $R^*R(L^1)$. A thorough treatment of the problem is given in [BFP, Theorem 5.2], by means of the spherical Fourier transform and a Paley-Wiener theorem. These tools are peculiar to the homogeneous setting, but turn out not to be necessary for the core of the argument, which might possibly be adapted to the general case.

Since $R^*Rf = f*\Psi$, let us first look for a function Φ which is a right convolution inverse for Ψ, i.e., is such that $\Psi * \Phi = \delta_v$. Incidentally, observe that $\Psi \geq 0$, whence Φ must take values of both signs, and the identity above is the result of cancellations. The series involved, however, converges absolutely, which we see once we prove that $\Phi \in L^{1+\epsilon}$. We, of course, make Φ vanish on vertices odd distances from v. It is easy (though tedious) to obtain

$$\Phi(w) = \begin{cases} \dfrac{3q+1}{2(q+1)} & \text{if } v = w, \\ 0 & \text{if } d(v,w) \text{ is odd}, \\ -\dfrac{q(q-1)}{2(q+1)q^{2n}} & \text{if } d(v,w) = 2n > 0; \end{cases}$$

as anticipated, $\Phi \in L^{1+\epsilon}$ for $\epsilon > 0$. Now, the convolution of $f \in L^1$, $\Phi \in L^{1+\epsilon}$, $\Psi \in L^{2+\epsilon}$ is associative, being given by absolutely convergent series: this yields the desired inversion formula

$$R^*Rf * \Phi = (f * \Psi) * \Phi = f * (\Psi * \Phi) = f * \delta_v = f.$$

It would be nice to extend this 'geometric' approach to the non-homogeneous setting. Closer to the spirit of integral geometry, we should try to reconstruct the function f at each vertex v from its transform $\phi = Rf$ by a two-step algorithm, in analogy with the approach of [BFP] in the homogeneous case and with the method developed in [BCCP] to invert the X-ray transform. The first step would consist of integrating ϕ over the set of horocycles through v, that is, of applying the dual operator R^*. The result is the function $g = R^*Rf = \sum_{w\in T} \psi(\,\cdot\,, w)\, f(w)$. The second step would consist of recovering f from g, that is, of inverting the summation operator with kernel ψ by means of the summation operator with some kernel ξ, which would thus have to satisfy

$$\sum_{v\in T} \xi(u,v)\,\psi(v,\,\cdot\,) = \delta_u \quad \text{for every } u \in T.$$

We cannot presently determine such ξ in the general non-homogeneous case. We shall provide in the next section, however, a different inversion formula, which does not factor through the action of R^*, yet whose expression in the homogeneous case is intriguingly simple.

6. An explicit inversion formula for the Radon transform

We shall prove a recursive inversion formula for R which is not geometric in the sense of the previous sections. This formula expresses the value of Rf at a vertex v as a linear combination of suitable averages of Rf over the sets of horocycles which pass through the generic vertex w and are contained in the sector $S(v, w)$.

The coefficients of this linear combination grow fast; we shall nevertheless be able to establish the validity of our inversion formula for all $f \in L^1(T)$.

As usual, let v be a fixed vertex, and, if $w \in T$ is at even distance $2n$ from v, we denote the finite path from v to w by $[v_0{=}v, v_1, \ldots, v_{2n}{=}w]$. Recall that each vertex of T has at least three neighbors.

THEOREM 6.1: *The following inversion formula holds for $f \in L^1(T)$:*

$$f(v) = \sum_{w:\ d(v,w)\ \text{even}} A_w \int_{\mathcal{H}_w^v} Rf(h)\, d\mu_v(h),$$

where A_w, for even $d(v,w)$, is given recursively by

$$A_w = \begin{cases} 1 & \text{if } w = v, \\ \displaystyle\sum_{k=0}^{n-1} A_{v_{2k}}(1 - q_{v_{n+k}}) \prod_{j=k+1}^{n-1} q_{v_{n+j}} & \text{if } w \neq v. \end{cases}$$

Remark 6.2: Assume $d(v,w)$ is even. Setting

$$B_w = \begin{cases} 1 & \text{if } w = v, \\ A_w - A_{v_{2n-2}} & \text{if } w \neq v, \end{cases}$$

that is, $A_w = \sum_{k=0}^n B_{v_{2k}}$, we get a simplified recursive expression:

$$B_w = \begin{cases} 1 & \text{if } w = v, \\ \displaystyle -\sum_{k=0}^{n-1} B_{v_{2k}} \prod_{j=k}^{n-1} q_{v_{n+j}} & \text{if } w \neq v. \end{cases}$$

If $\alpha \in \{0,1\}^n$ is such that $\alpha_n = 0$, and if $1 \le j \le n$, set

$$\ell(j,\alpha) = j - 1 + \min_{\substack{i \ge j \\ \alpha_i = 0}} i.$$

In terms of this notation we get

$$B_w = \begin{cases} 1 & \text{if } w = v, \\ \displaystyle (-1)^n \sum_{\substack{\alpha \in \{0,1\}^n \\ \text{with } \alpha_n = 0}} (-1)^{|\alpha|} \prod_{j=1}^n q_{v_{\ell(j,\alpha)}} & \text{if } w \neq v, \end{cases}$$

and thus A_w is given in closed form by

$$A_w = 1 + \sum_{k=1}^n (-1)^k \sum_{\substack{\alpha \in \{0,1\}^k \\ \text{with } \alpha_k = 0}} (-1)^{|\alpha|} \prod_{j=1}^k q_{v_{\ell(j,\alpha)}}.$$

The first few $A_w = A_{v_{2n}}$ are (setting $q_j = q_{v_j}$ for brevity):

$$\begin{aligned}
A_{v_0} &= 1,\\
A_{v_2} &= 1 - q_1,\\
A_{v_4} &= 1 - q_1 + q_1q_3 - q_2q_3,\\
A_{v_6} &= 1 - q_1 + q_1q_3 - q_2q_3 - q_1q_3q_5 + q_2q_3q_5 + q_1q_4q_5 - q_3q_4q_5,\\
A_{v_8} &= 1 - q_1 + q_1q_3 - q_2q_3 - q_1q_3q_5 + q_2q_3q_5 + q_1q_4q_5 - q_3q_4q_5\\
&\qquad + q_1q_3q_5q_7 - q_2q_3q_5q_7 - q_1q_4q_5q_7 + q_3q_4q_5q_7\\
&\qquad - q_1q_3q_6q_7 + q_2q_3q_6q_7 + q_1q_5q_6q_7 - q_4q_5q_6q_7
\end{aligned}$$

(note that A_w does not depend on q_v or q_w).

LEMMA 6.3: *A necessary condition for $\mu_v(\mathcal{H}_u^v \cap \mathcal{H}_w)$ to be nonzero is that $d(v,u)$ and $d(v,w)$ be both even or both odd. More precisely, if $d(v,w) = 2n$ we have*

$$\mu_v(\mathcal{H}_u^v \cap \mathcal{H}_w) = \begin{cases} \mu_v(\mathcal{H}_v) = 1 & \text{if } u = w = v,\\ \mu_v(\mathcal{H}_w^v) = \dfrac{1}{q_{v_0}+1} \displaystyle\prod_{j=1}^{2n-1} \frac{1}{q_{v_j}} & \text{if } u = w \neq v,\\ \mu_v(\mathcal{H}_{v_{2k}} \cap \mathcal{H}_w) = \left(1 - \dfrac{1}{q_{v_{n+k}}}\right) \dfrac{1}{q_{v_0}+1} \displaystyle\prod_{j=1}^{n+k-1} \frac{1}{q_{v_j}} & \text{if } u = v_{2k} \neq w,\\ \mu_v(\emptyset) = 0 & \text{otherwise.} \end{cases}$$

This can be proved by a straightforward verification.

Proof of Theorem 6.1: First observe that R is continuous from $L^1(T)$ to $L^\infty(\mathcal{H})$ (in fact $\|R\| = 1$). So

$$Rf = R\Big(\sum_{w\in T} f(w)\delta_w\Big) = \sum_{w\in T} f(w)R\delta_w = \sum_{w\in T} f(w)\chi_{\mathcal{H}_w}.$$

Let $u \in T$. Since $\mu_v(\mathcal{H}_u^v \cap \mathcal{H}_w) \le \mu_v(\mathcal{H}_w) = 1$ for every vertex w, if $f \in L^1$ we have

$$\sum_{w\in T} \int_{\mathcal{H}_u^v} |f(w)|\, \chi_{\mathcal{H}_w}\, d\mu_v = \sum_{w\in T} |f(w)|\, \mu_v(\mathcal{H}_u^v \cap \mathcal{H}_w) < \infty;$$

therefore by absolute convergence

$$\begin{aligned}
\int_{\mathcal{H}_u^v} Rf\, d\mu_v &= \int_{\mathcal{H}_u^v} \sum_{w\in T} f(w)\chi_{\mathcal{H}_w}\, d\mu_v = \sum_{w\in T} f(w) \int_{\mathcal{H}_u^v} \chi_{\mathcal{H}_w}\, d\mu_v\\
&= \sum_{w\in T} f(w)\mu_v(\mathcal{H}_u^v \cap \mathcal{H}_w).
\end{aligned}$$

If $d(v,u)$ is even, by Lemma 6.3 the last summation is only taken over those vertices $w \in S(v,u)$ which are at even distance from v.

Hence

$$\sum_{u:\, d(v,u)\text{ even}} A_u \int_{\mathcal{H}_u^v} Rf\, d\mu_v = \sum_{u:\, d(v,u)\text{ even}} A_u \sum_{\substack{w\in S(v,u):\\ d(v,w)\text{ even}}} f(w)\mu_v(\mathcal{H}_u^v \cap \mathcal{H}_w)$$
$$= \sum_{w:\, d(v,w)\text{ even}} f(w) \sum_{m=0}^{n} A_{v_{2m}} \mu_v(\mathcal{H}_{v_{2m}}^v \cap \mathcal{H}_w).$$

Of course we must justify the second equality in this last formula. But after doing that, the inversion formula will be completely proved: making use of Lemma 6.3 and the recursive definition of A_u, an easy computation shows that for w an even distance from v

$$\sum_{m=0}^{n} A_{v_{2m}} \mu_v(\mathcal{H}_{v_{2m}}^v \cap \mathcal{H}_w) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

The required justification again comes from absolute convergence of the series. Fix w at an even distance from v. For $0 \le m \le n$, observe that $q_u \ge 2$ for every u, and that, whenever $1 \le k \le m$ and $\alpha \in \{0,1\}^k$ is such that $\alpha_k = 0$, we have $\ell(j,\alpha) < \ell(j',\alpha)$ if $1 \le j < j' \le k$. From the closed expression of A_u in Remark 6.2 we get

$$|A_{v_{2m}}| \prod_{i=0}^{n+m-1} \frac{1}{q_{v_i}} \le \prod_{i=0}^{n+m-1} \frac{1}{q_{v_i}} + \sum_{k=1}^{m} \sum_{\substack{\alpha\in\{0,1\}^k\\ \text{with } \alpha_k=0}} \prod_{\substack{1\le i\le k\\ i\ne \ell(1,\alpha),\ldots,\ell(k,\alpha)}} \frac{1}{q_{v_i}}$$
$$\le 2^{-n-m} + \sum_{k=1}^{m} 2^{k-1} 2^{-n-m+k} \le 2^{1-n+m}.$$

Probably this estimate, or an equivalent one, can also be derived from the recursive expression of A_u. From this formula and from Lemma 6.3 we get

$$\sum_{m=0}^{n} |A_{v_{2m}}| \mu_v(\mathcal{H}_{v_{2m}}^v \cap \mathcal{H}_w) \le \sum_{m=0}^{n} |A_{v_{2m}}| \prod_{i=0}^{n+m-1} \frac{1}{q_{v_i}} \le \sum_{m=0}^{n} 2^{1-n+m} \le 4.$$

Therefore

$$\sum_{w:\, d(v,w)\text{ even}} \sum_{m=0}^{n} |f(w)A_{v_{2m}}| \mu_v(\mathcal{H}_{v_{2m}}^v \cap \mathcal{H}_w) \le \sum_{w:\, d(v,w)\text{ even}} 4\,|f(w)| < \infty. \qquad \blacksquare$$

In the homogeneous case we simply have

$$A_w = \begin{cases} 1 & \text{if } w = v, \\ 1-q & \text{if } w \neq v. \end{cases}$$

Setting $\mathcal{H}_{(v)} = \{h_v(2n,\omega) \colon n > 0,\ \omega \in \Omega\}$, the inversion formula becomes

$$f(v) = \left[\int_{\mathcal{H}_v} + (1-q)\int_{\mathcal{H}_{(v)}}\right] Rf(h)\, d\mu_v(h)$$

(observe that $\mu_v(\mathcal{H}_{(v)}) = \infty$).

If T is semi-homogeneous, that is, if

$$q_u = \begin{cases} q & \text{if } d(v,u) \text{ is even}, \\ p & \text{if } d(v,u) \text{ is odd}, \end{cases}$$

then

$$A_w = \begin{cases} 1 & \text{if } w = v, \\ 1 - p + (p-q)\,\dfrac{p + (-1)^n p^n}{p+1} & \text{if } w \neq v. \end{cases}$$

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